Summary of key points

1 The **addition** (or compound-angle) formulae are:

- $\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$ $\quad\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$
- $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$ $\quad\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$
- $\tan(A + B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ $\quad\tan(A - B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

2 The **double-angle** formulae are:

- $\sin 2A \equiv 2 \sin A \cos A$
- $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
- $\tan 2A \equiv \dfrac{2 \tan A}{1 - \tan^2 A}$

3 For positive values of a and b,

- $a \sin x \pm b \cos x$ can be expressed in the form $R \sin(x \pm \alpha)$
- $a \cos x \pm b \sin x$ can be expressed in the form $R \cos(x \mp \alpha)$

with $R > 0$ and $0 < \alpha < 90°$ $\left(\text{or } \dfrac{\pi}{2}\right)$

where $R \cos \alpha = a$ and $R \sin \alpha = b$ and $R = \sqrt{a^2 + b^2}$.

Addition formulae

These identities will appear in the A-level section of your formulae booklet :

$$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Each of these is actually **two** different identities. Be careful with ± and ∓. Take the top sign of each pair for one identity, and the bottom sign for the other.

$\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$
$\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$

$\cos x = \sin(90° - x)$ so $\cos 70° = \sin 20°$ and $\cos 20° = \sin 70°$. The answer contains $\tan 20°$, so write everything in terms of $\sin 20°$ and $\cos 20°$ before rearranging.

Worked example

Given that $2\cos(x + 70°) = \sin(x + 20°)$, show, without using a calculator, that $\tan x = \frac{1}{3}\tan 20°$ **(4 marks)**

$2(\cos x \cos 70° - \sin x \sin 70°) = \sin x \cos 20° + \cos x \sin 20°$

$2\cos x \sin 20° - 2\sin x \cos 20° = \sin x \cos 20° + \cos x \sin 20°$

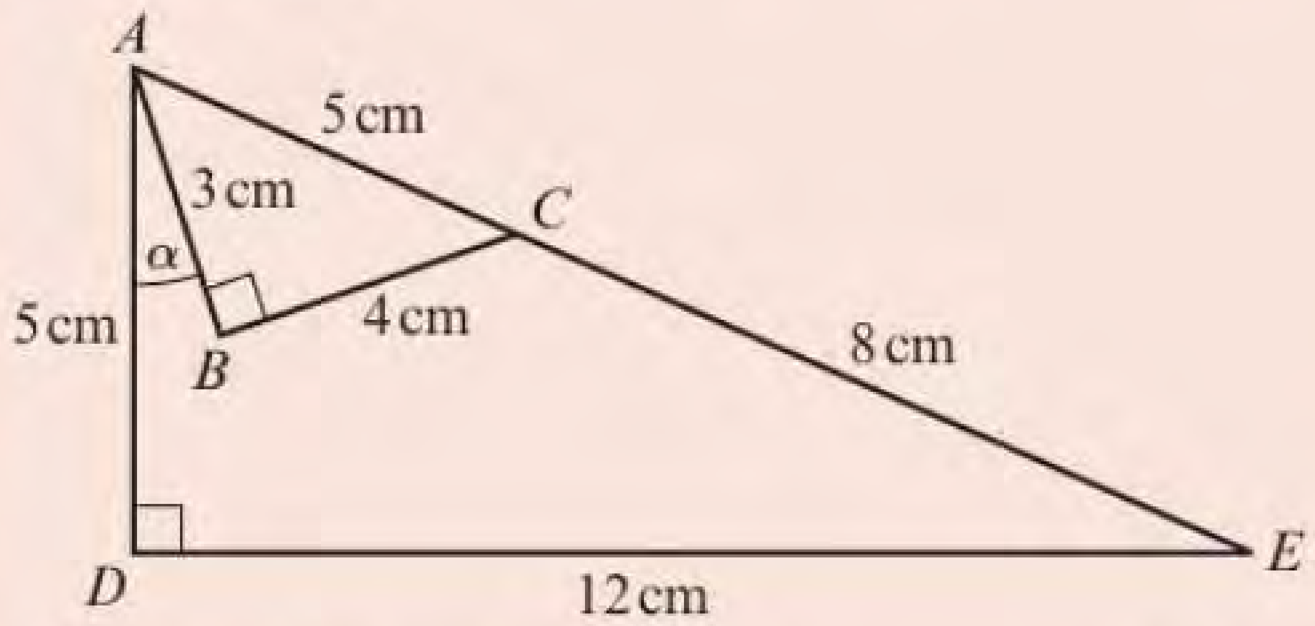
$\cos x \sin 20° = 3\sin x \cos 20°$

$\frac{\sin 20°}{\cos 20°} = \frac{3\sin x}{\cos x}$

$\tan 20° = 3\tan x$ so $\tan x = \frac{1}{3}\tan 20°$

Worked example

The diagram shows two right-angled triangles ABC and ADE.



(a) Show that $\cos\alpha = \frac{63}{65}$ **(3 marks)**

$\cos\alpha = \cos(\angle EAD - \angle CAB)$
$= \cos\angle EAD \cos\angle CAB + \sin\angle EAD \sin\angle CAB$
$= \frac{5}{13} \times \frac{3}{5} + \frac{12}{13} \times \frac{4}{5}$
$= \frac{15}{65} + \frac{48}{65} = \frac{63}{65}$

(b) Hence, or otherwise, find the exact value of the length of BD. **(3 marks)**

$BD^2 = AB^2 + AD^2 - 2 \times AB \times AD \times \cos\alpha$
$= 3^2 + 5^2 - 2 \times 3 \times 5 \times \frac{63}{65}$
$= \frac{64}{13}$

$BD = \sqrt{\frac{64}{13}} = \frac{8}{\sqrt{13}}$

Be careful when applying the addition formulae for cos. The signs are **swapped** on the right-hand side.

Hence or otherwise

If a question says 'hence or otherwise' it means you can use the earlier part of the question to help you find your answer. This is usually easier than starting this part of the question from scratch.

Use the formula for $\cos(A - B)$, then the **cosine rule** to find the missing length.

Now try this

1 By writing $\sin 15° = \sin(45° - 30°)$, show that $\operatorname{cosec} 15° = \frac{4}{\sqrt{6} - \sqrt{2}}$ **(4 marks)** Answer

2 (a) Show that $\cos(x + 45°) = \frac{\cos x - \sin x}{\sqrt{2}}$ Answer **(3 marks)**

(b) Hence solve, for $0 \leqslant x \leqslant 360°$, $\cos x - \sin x = 0.5$
Give your answers in degrees to 1 decimal place. Answer **(6 marks)**

Had a look ☐ Nearly there ☐ Nailed it! ☐

Double angle formulae

These double angle formulae can be **derived** from the addition formulae. But they're really useful and you should **learn** them for your exam.

1 $\sin 2A \equiv 2\sin A\cos A$

2 $\cos 2A \equiv \cos^2 A - \sin^2 A$
$\equiv 2\cos^2 A - 1$
$\equiv 1 - 2\sin^2 A$

3 $\tan 2A \equiv \dfrac{2\tan A}{1 - \tan^2 A}$

You should learn all three versions of this one.

Worked example

Use the identity
$\cos(A + B) \equiv \cos A\cos B - \sin A\sin B$
to show that
$\cos 2A \equiv 1 - 2\sin^2 A$ **(2 marks)**

$\cos 2A \equiv \cos(A + A) \equiv \cos A\cos A - \sin A\sin A$
$\equiv \cos^2 A - \sin^2 A$
$\equiv (1 - \sin^2 A) - \sin^2 A$
$\equiv 1 - 2\sin^2 A$

You need to be able to prove all of the identities above using the addition formulae on page 82. You will need to use the identity $\sin^2 A + \cos^2 A \equiv 1$ to get your expressions completely in terms of $\sin^2$ or $\cos^2$.

Worked example

(a) By writing $\sin 3\theta$ as $\sin(2\theta + \theta)$, show that
$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ **(5 marks)**

$\sin 3\theta = \sin(2\theta + \theta)$
$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$
$= (2\sin\theta\cos\theta)\cos\theta + (1 - 2\sin^2\theta)\sin\theta$
$= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$
$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta$
$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$
$= 3\sin\theta - 4\sin^3\theta$

(b) Given that $\sin\theta = \frac{\sqrt{3}}{4}$, find the exact value of $\sin 3\theta$. **(2 marks)**

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
$= 3\left(\frac{\sqrt{3}}{4}\right) - 4\left(\frac{\sqrt{3}}{4}\right)^3$
$= \frac{3\sqrt{3}}{4} - \frac{3\sqrt{3}}{16}$
$= \frac{9\sqrt{3}}{16}$

As long as you aren't asked to prove a double angle formula in an exam question, it's OK to use it without deriving it. It's a good idea to show the steps when you use each identity clearly. You can use brackets to show the substitutions, or write down the identities you are using.

Working backwards

Identities can be applied in **both directions**. Make sure you are familiar enough with each of the double angle formulae to recognise **either side** when you see it.

Here are some useful variations:

- $\sin A\cos A = \frac{1}{2}\sin 2A$
- $\cos A + 1 = 2\cos^2\frac{1}{2}A$
- $1 - \cos A = 2\sin^2\frac{1}{2}A$

Part (b) says 'hence', so use the result from part (a) to get started. Look for similarities between the expression in part (a) and the equation in part (b).

Now try this

1 Use the identity
$\cos(A + B) \equiv \cos A\cos B - \sin A\sin B$
to show that $\cos 2A \equiv 2\cos^2 A - 1$ **(2 marks)**

Answer

2 (a) Show that $\cos 3x = 4\cos^3 x - 3\cos x$

Answer **(5 marks)**

(b) Hence solve, for $0 \leqslant x \leqslant 90°$,
$8\cos^3 x - 6\cos x - 1 = 0$ **(3 marks)**

Answer

$a\cos\theta \pm b\sin\theta$

You can use the addition formulae to write expressions of the form $a\cos\theta \pm b\sin\theta$ in the form $R\cos(\theta \mp \alpha)$ or $R\sin(\theta \pm \alpha)$. This can help you solve harder trig equations.

If you use these rules, be careful with the signs. You will usually be able to use **positive** values of a and b.

Golden rules

1. $a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha)$
2. $a\sin\theta \pm b\cos\theta = R\sin(\theta \pm \alpha)$

where $R = \sqrt{a^2 + b^2}$ and $\alpha = \arctan\left(\frac{b}{a}\right)$

Worked example

(a) Express $5\cos x - 3\sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$ **(4 marks)**

$R = \sqrt{5^2 + 3^2} = \sqrt{34}$

$\alpha = \arctan\left(\frac{3}{5}\right) = 0.5404...$

$5\cos x - 3\sin x = \sqrt{34}\cos(x + 0.5404...)$

(b) Hence, or otherwise, solve the equation

$5\cos x - 3\sin x = 4$

for $0 \leqslant x \leqslant 2\pi$, giving your answers to 2 decimal places. **(5 marks)**

$\sqrt{34}\cos(x + 0.5404...) = 4$

$\cos(x + 0.5404...) = \frac{4}{\sqrt{34}}$

$x + 0.5404... = 0.8148...$

$x = 0.8148... - 0.5404...$

$= 0.2744...$

or $x + 0.5404... = 2\pi - 0.8148...$

$= 5.4683...$

$x = 5.4683... - 0.5404...$

$= 4.9279...$

$x = 0.27, 4.93$ (2 d.p.)

If you don't want to learn the rules above, you can use the addition formulae from page 82:

$5\cos x - 3\sin x = R\cos x\cos\alpha - R\sin x\sin\alpha$

Equate coefficients of $\cos x$: $5 = R\cos\alpha$ ①

Equate coefficients of $\sin x$: $3 = R\sin\alpha$ ②

①^2 + ②^2: $R^2(\cos^2\alpha + \sin^2\alpha) = 5^2 + 3^2$

$R = \sqrt{5^2 + 3^2}$

② ÷ ①: $\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{3}{5}$

$\alpha = \arctan\left(\frac{3}{5}\right)$

Be careful. In part (b) of this question, the principal value from your calculator will not give you an answer in the required range.

Problem solved!

Pay attention to these three Rs when solving trig equations in your exam:

- **Rounding** – Don't round any values until the end of your calculation. Learn how to use the 'STORE' or 'MEMORY' functions on your calculator so you can use unrounded values, or write down values to at least 4 decimal places.
- **Radians** – Look at the range to decide whether you should be working in degrees or radians. Make sure your calculator is in the correct mode.
- **Range** – Check that all your solutions are within the specified range, and check that you have found every possible solution within that range.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Now try this

1 (a) Express $3\sin 2\theta + 2\cos 2\theta$ in the form $R\sin(2\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$ Answer **(4 marks)**

(b) Hence, or otherwise, solve the equation

$3\sin 2\theta + 2\cos 2\theta + 1 = 0$

for $0 \leqslant \theta \leqslant \pi$, giving your answers to 2 decimal places. **(5 marks)** Answer

2 The function f is defined by

$y = \text{f} : x \mapsto \sqrt{3}\cos x + \sin x$

(a) Given that $\text{f}(x) = R\cos(x - \alpha)$, where $R > 0$ and $0 < \alpha < 90°$, find the value of R and the value of α. Answer **(4 marks)**

(b) Hence sketch the graph of $y = \text{f}(x)$ for $0 \leqslant x \leqslant 360°$, showing clearly the coordinates of any maxima or minima, and any points where the graph meets the coordinate axes. Answer **(5 marks)**