Iterative Methods

Without further ado, let's have a look at how to home in on those roots of functions using iteration formulas. Iteration is like turbo-charged trial and improvement, so grab your calculator and prepare to write down a lot of digits.

Use an Iteration Formula to find Approximations of Roots

Some equations are just too darn tricky to **solve properly**. For these, you need to find **approximations** to the roots, to a certain level of **accuracy**. You'll usually be told the value of x that a root is close to, or the interval that it lies in (as on the previous pages), and then **iteration** does the rest.

Iteration works like this: you put an approximate value of a root x into an **iteration formula**, and out pops a slightly more accurate value. Then you **repeat** as necessary until you have an **accurate enough** answer.

Example: Use the **iteration formula** $x_{n+1} = \sqrt[3]{x_n + 4}$ to solve $x^3 - 4 - x = 0$ to 2 d.p. Start with $x_0 = 2$.

1) The notation x_n just means the **approximation of x at the n^th^ iteration**.
So putting x_0 in the formula for x_n gives you x_{n+1}, which is x_1, the first iteration.

2) $x_0 = 2$, so $x_1 = \sqrt[3]{x_0 + 4} = \sqrt[3]{2 + 4} = \mathbf{1.8171...}$ ← Leave this in your calculator for accuracy.

3) This value now gets put back into the formula to find x_2:
$x_1 = 1.8171...$, so $x_2 = \sqrt[3]{x_1 + 4} = \sqrt[3]{1.8171... + 4} = \mathbf{1.7984...}$ ← You should now just be able to type '$\sqrt[3]{(\text{ANS} + 4)}$' in your calculator and keep pressing enter for each iteration.

4) Carry on until you get answers that are the **same** when **rounded to 2 d.p**:
$x_2 = 1.7984...$, so $x_3 = \sqrt[3]{x_2 + 4} = \sqrt[3]{1.7984... + 4} = \mathbf{1.7965...}$

5) x_2, x_3, and all further iterations are the same when rounded to 2 d.p., so the root is $\mathbf{x = 1.80}$ **to 2 d.p.**

Rearrange the Equation to get the Iteration Formula

The iteration formula is just a **rearrangement** of the equation, leaving a **single** 'x' on one side.

There are often lots of **different ways** to rearrange the equation, so in the exam you'll probably be asked to '**show that**' it can be rearranged in a certain way, rather than starting from scratch.

You can also rearrange $x^3 - x^2 - 9 = 0$ into the iteration formula $x_{n+1} = \sqrt{x_n^3 - 9}$, which behaves differently, as shown below.

Example: Show that $x^3 - x^2 - 9 = 0$ can be rearranged into $x = \sqrt{\frac{9}{x-1}}$.

The '9' is on its own in the fraction so try: → $x^3 - x^2 - 9 = 0$
$\Rightarrow x^3 - x^2 = 9$

The LHS can be factorised now: → $x^2(x - 1) = 9$

Get the x^2 on its own by dividing by $x - 1$: → $x^2 = \frac{9}{x-1}$

Finally take the square root of both sides: → $x = \sqrt{\frac{9}{x-1}}$

You can now use the **iteration formula** $x_{n+1} = \sqrt{\frac{9}{x_n - 1}}$ to find approximations of the roots.

Sometimes an iteration formula just will not find a root. In these cases, no matter how close to the root you have x_0, the iteration sequence **diverges** — the numbers get further and further apart. The iteration might also **stop working** — e.g. if you have to take the **square root** of a **negative number**.

Example: The equation $x^3 - x^2 - 9 = 0$ has a root close to $x = 2.5$.
What is the result of using $x_{n+1} = \sqrt{x_n^3 - 9}$ with $x_0 = 2.5$ to find this root?

Start with $x_1 = \sqrt{2.5^3 - 9} = \mathbf{2.5739...}$ (seems okay so far...)

Subsequent iterations give: $x_2 = \mathbf{2.8376...}$, $x_3 = \mathbf{3.7214...}$, $x_4 = \mathbf{6.5221...}$ — so the sequence **diverges**.

An iteration formula $x_{n+1} = f(x_n)$ will converge to a root a if the following two conditions are satisfied:

1) The starting value x_0 is **sufficiently close** to a
2) The **derivative** of f(x) is **small** at a, i.e. $|\mathbf{f'(a)}| < 1$.

See p.130 for more about what this looks like.

More on Iterative Methods

Whilst iterations are relatively thrilling on their own, put them into a diagram and the world is your lobster. Well, that might be a bit of an exaggeration, but you can show convergence or divergence easily on a pretty diagram.

You can show **Iterations** on a **Diagram**

Once you've calculated a **sequence of iterations** using $x_{n+1} = f(x_n)$, you can plot the points on a **diagram** and use it to show whether your sequence **converges** or **diverges**.

Sketching Iterations

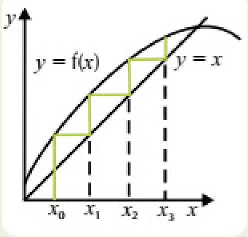
1) First, sketch the graphs of $y = x$ and $y = f(x)$ (where f(x) is the iterative formula). The point where the **two graphs meet** is the root you're aiming for.
2) Draw a **vertical line** from the x-value of your starting point (x_0) until it meets the curve $y = f(x)$.
3) Now draw a **horizontal line** from this point to the line $y = x$. At this point, the x-value is x_1, the value of your first iteration. This is **one step**.
4) Draw **another step** — a vertical line from this point to the curve, and a horizontal line joining it to the line $y = x$.
5) **Repeat step 4)** for each of your iterations.
6) If your steps are getting **closer and closer to the root**, the sequence of iterations is **converging**. If the steps are moving **further and further away** from the root, the sequence is **diverging**.

This method produces two different types of diagrams — cobweb diagrams and staircase diagrams.

Convergent *iterations* ***Home In*** *on the* ***Root***

It's probably easiest to follow the method by looking at a few **examples**:

1



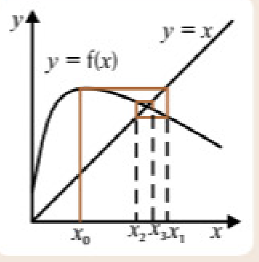
This is an example of a **convergent staircase diagram**. Starting at x_0, the next iterations x_1, x_2 and x_3 are getting **closer** to the point where the two graphs intersect (the root).

In each case, the diagram will look different depending on where your starting point is.

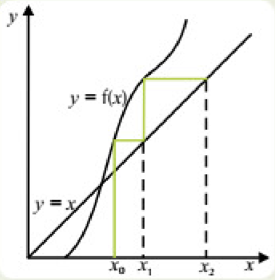
In general, these diagrams will converge if your starting value x_0 is close enough to the root, and if the graph of f(x) isn't too steep at the root (the gradient f′(x) needs to be between –1 and 1) — these conditions were on p.128.

2

This is an example of a **convergent cobweb diagram**. In this case, the iterations **alternate** between being **below** the root and **above** the root, but are still getting **closer** each time.



3



This is an example of a **divergent staircase diagram**. Starting at x_0, the iterations x_1 and x_2 are getting **further away** from the root.

After hours of futile iteration, Todd realised his staircase was divergent.

More on Iterative Methods

Exam Questions combine all the Different Methods

In exam questions, you might have to use **different methods** to find roots — like in this giant worked example.

Example: The graph below shows both roots of the continuous function $f(x) = 6x - x^2 + 13$.

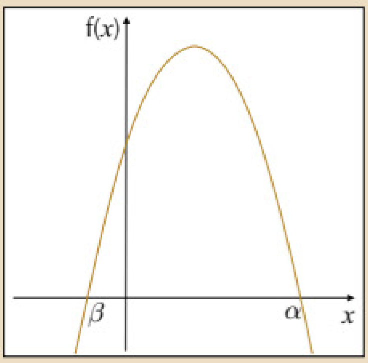
a) Show that the positive root, α, of $f(x) = 0$ lies in the interval $7 < x < 8$.

b) Show that $6x - x^2 + 13 = 0$ can be rearranged into the formula $x = \sqrt{6x + 13}$.

c) Use the iteration formula $x_{n+1} = \sqrt{6x_n + 13}$ and $x_0 = 7$ to find α to 1 d.p.

d) Sketch a diagram to show the convergence of the sequence for x_1, x_2 and x_3.

e) Use the Newton-Raphson method to find the negative root, β, to 5 s.f. Start with $x_0 = -1$. Use bounds to check your root.



a) $f(x)$ is a **continuous function**, so if $f(7)$ and $f(8)$ have **different signs** then there is a root in the interval $7 < x < 8$:

$f(7) = (6 \times 7) - 7^2 + 13 = 6$.

$f(8) = (6 \times 8) - 8^2 + 13 = -3$.

There is a **change of sign** so $7 < \alpha < 8$.

b) Get the x^2 on its own to make: $6x + 13 = x^2$

Now take the (positive) square root to leave: $x = \sqrt{6x + 13}$

c) Using $x_{n+1} = \sqrt{6x_n + 13}$ with $x_0 = 7$, gives $x_1 = \sqrt{6 \times 7 + 13} = 7.4161...$

Continuing the iterations:

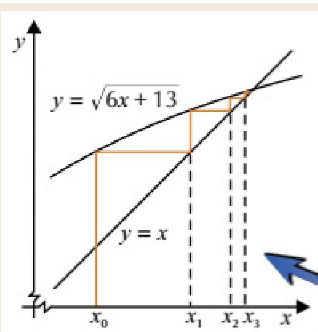
$x_2 = \sqrt{6 \times 7.4161... + 13} = 7.5826...$ $\quad x_3 = \sqrt{6 \times 7.5826... + 13} = 7.6482...$

$x_4 = \sqrt{6 \times 7.6482... + 13} = 7.6739...$ $\quad x_5 = \sqrt{6 \times 7.6739... + 13} = 7.6839...$

$x_6 = \sqrt{6 \times 7.6839... + 13} = 7.6879...$ $\quad x_7 = \sqrt{6 \times 7.6879... + 13} = 7.6894...$

x_4 to x_7 all round to **7.7 to 1 d.p.**, so to 1 d.p. $\alpha = 7.7$

The list of results from each iteration x_1, x_2, x_3... is called the iteration sequence.



d) Sketch $y = \sqrt{6x + 13}$ and $y = x$ on the same axes, and mark on the position of x_0. All you have to do is draw on the **lines** and label the **values** of x_1, x_2 and x_3. You can see from the diagram that the sequence is a **convergent staircase**.

e) Find $f'(x)$: $f'(x) = 6 - 2x$.

Putting this into the **Newton-Raphson formula** gives: $x_{n+1} = x_n - \dfrac{6x_n - x_n^2 + 13}{6 - 2x_n}$

Starting with $x_0 = -1$, this gives $x_1 = -1 - \dfrac{6(-1) - (-1)^2 + 13}{6 - 2(-1)} = -1.75$

Further iterations give $x_2 = -1.690789...$

$x_3 = -1.690415...$

$x_4 = -1.690415...$ So **$\beta = -1.6904$ to 5 s.f.**

The **upper and lower bounds** for this are -1.69035 and -1.69045, and $f(-1.69035) = 0.0006$, $f(-1.69045) = -0.0003$.

There is a **sign change** so the root is accurate to 5 s.f.

You might have to **compare** the different iteration methods — to do this, all you have to do is use each method to find the same root, then think about which one was the **easiest to use**, which was the **quickest** (i.e. took the fewest iterations), etc.