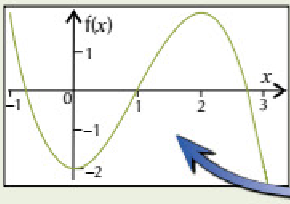
Location of Roots

Time to go back to your roots — and by 'roots' I mean, 'values of x for which f(x) = 0'. When all those lovely algebraic methods you've learnt just won't do the trick, these numerical methods should get you close enough.

A **Change of Sign** from **f(a) to f(b)** means a **Root Between a and b**

You may be asked to '**solve**' or '**find the roots of**' an equation (where **f(*x*) = 0**). This is **exactly the same** as finding the **value of *x*** where the graph **crosses** (or just **touches**) the ***x*-axis**. The **graph** of the function gives you a rough idea **how many** roots there are (**if any**) and **where**.



E.g. the function $f(x) = 3x^2 - x^3 - 2$ (shown here) has **3 roots** in the interval $-1 \le x \le 3$, since it crosses the *x*-axis **three times** (i.e. there are 3 solutions to the equation $3x^2 - x^3 - 2 = 0$).

You can also see from the graph that $x = 1$ is a root, and the other roots are **close to $x = -1$ and $x = 3$**.

Look at the graph above at the root $x = 1$. For *x*-values **just before** the root, f(*x*) is **negative**, and **just after** the root, f(*x*) is **positive**. It's the other way around for the other two roots, but either way:

if f(*x*) **changes sign**, then you know it has passed through a root.

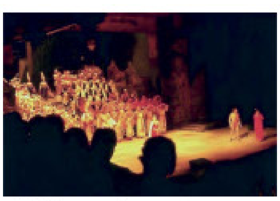
This is only true for functions that are **continuous** over the interval you're looking at — ones that are **joined up** all the way along with no 'jumps' or gaps. If not, the sign change might be because of something other than the root.

f(x) = tan x is an example of a non-continuous function — it has gaps where f(x) changes sign even though there's no root (see p.60).

To show that a root lies in the interval between two values 'a' and 'b':

1) Find **f(a)** and **f(b)**.
2) If the two answers have **different signs**, and the function is **continuous** over that interval, there's a root somewhere between 'em.

You can have roots where f(x) doesn't change sign. This happens if the graph of f(x) just touches the x-axis, rather than passing through it.



"You're going to show me *what* in the interval?!"

Example: Show that $x^4 + 3x - 5 = 0$ has a root in the interval $1.1 \le x \le 1.2$.

1) Put both 1.1 and 1.2 into the expression: $f(1.1) = (1.1)^4 + (3 \times 1.1) - 5 = \mathbf{-0.2359}$

$f(1.2) = (1.2)^4 + (3 \times 1.2) - 5 = \mathbf{0.6736}$

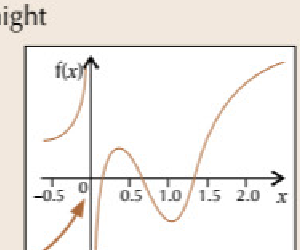
2) f(1.1) and f(1.2) have **different signs**, and f(*x*) is **continuous**, so there's a root in the interval [1.1, 1.2].

Make sure you're familiar with the different notation for closed and open intervals: [1.1, 1.2] means $1.1 \le x \le 1.2$, (1.1, 1.2) means $1.1 < x < 1.2$, and (1.1, 1.2] means $1.1 < x \le 1.2$.

A **Large Interval** may **Hide Roots**

You need to be careful that the interval you're looking at is not **too large**. If it's too big, there could be an **even number of roots** within the interval — so the sign would **not** appear to change and you'd miss those roots.

- For the function shown here, f(0.5) and f(1.5) are both **positive**, so you might **incorrectly** assume that there are **no roots** in the interval $0.5 \le x \le 1.5$.
- But the graph shows that there are actually **two roots** — one in the interval [0.5, 1.0] and another in the interval [1.0, 1.5]. So f(*x*) changes from positive to negative then **back to positive again**.
- You can also see here that although f(*x*) **changes sign** before and after $x = 0$, this is **not caused by a root** — the function is **not continuous** over this interval as there is an **asymptote** at $x = 0$.



Similarly, if there were **3** roots in an interval, the sign might change from positive to negative to positive to negative — you might make the mistake of thinking there was only one root in the interval, when in fact there were 3.

Location of Roots

*Use **Upper and Lower Bounds** to '**Show that**' a root is correct*

You might be given an approximation to a root and be asked to **show** that it's correct to a certain **accuracy**. This is like showing that the root lies in a certain interval — the trick is to work out the **right interval**.

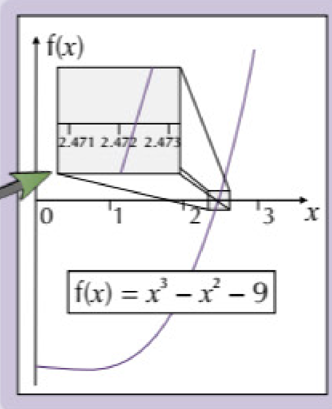
Example: Show that $x = 2.472$ is a root of the equation $x^3 - x^2 - 9 = 0$ to 3 d.p.

1) If $x = 2.472$ is a root rounded to 3 decimal places, the exact root must lie between the **upper and lower bounds** of this value — **2.4715** and **2.4725**. Any value in this interval would be rounded to 2.472 to 3 d.p.

2) The function $f(x) = x^3 - x^2 - 9$ is **continuous**, so you know the root lies in the interval $2.4715 \le x \le 2.4725$ if f(2.4715) and f(2.4725) have **different signs**.

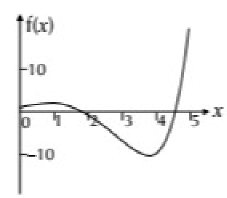
3) $f(2.4715) = 2.4715^3 - 2.4715^2 - 9 = \mathbf{-0.0116...}$
and $f(2.4725) = 2.4725^3 - 2.4725^2 - 9 = \mathbf{0.0017...}$

4) f(2.4715) and f(2.4725) have different signs, so there must be a root in between them. Since any value between would be rounded to 2.472 to 3 d.p. this answer **must be correct**.



Practice Questions

Q1 The graph shows the function $f(x) = e^x - x^3$ for $0 \le x \le 5$.
How many roots does the equation $e^x - x^3 = 0$ have in the interval $0 \le x \le 5$?



Q2 Show that there is a root in the interval:

a) (3, 4) for $\sin(2x) = 0$ (x in radians),

b) (2.1, 2.2) for $\ln(x - 2) + 2 = 0$,

c) [4.3, 4.5] for $x^3 - 4x^2 = 7$.

Q3 By selecting an appropriate interval, show that $x = 1.2$ is a root of the equation $x^3 + x - 3 = 0$ to 1 d.p.

Exam Questions

Q1 The graph of the function $f(x) = 2xe^x - 3$ crosses the x-axis at the point $P(p, 0)$.

a) Show that $0.7 < p < 0.8$. [2 marks]

b) Show that $p = 0.726$ to 3 d.p. [3 marks]

Q2 The function $g(x) = \operatorname{cosec} x - 2$.

a) Give two reasons why using a change of sign method to check for a root of $g(x)$ over the interval $[0, \pi]$ will fail. [2 marks]

b) Show that $x = 0.5$ is an approximation to one root of $g(x) = 0$ to 1 decimal place. [3 marks]

c) Solve $g(x) = 0$ to find an exact value for the first root in the interval $[0, \pi]$. [2 marks]

d) Show that $g(x)$ has a second root in the interval (2.6, 2.7). [2 marks]

The signs they are a-changin'...

You could come across these 'show that' questions nestled inside bigger ones on numerical methods. They might start by asking you to 'show that' a root is in an interval, then find it using iteration (coming up next), then 'show that' the answer you found was correct to a given accuracy. Just remember that the sign changes when the function passes through a root (as long as the function is continuous and the interval is small enough to capture a single root). Job done.