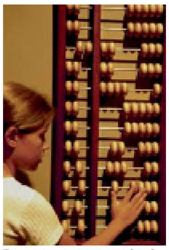
Addition and Double Angle Formulas

You might have noticed that there are quite a lot of formulas lurking in this here trigonometry jungle. There are some more coming up on these pages too I'm afraid, so brace yourself.

You can use the **Addition Formulas** to find **Sums of Angles**

You can use the **addition formulas** to find the sin, cos or tan of the **sum** or **difference** of two angles.

When you have an expression like sin $(x + 60°)$ or cos $\left(n - \frac{\pi}{2}\right)$, you can use these formulas to **expand the brackets**.



Fran activated the 'hard trig' setting on her new-fangled adding machine.

$$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

These formulas are given to you on the formula sheet.

Watch out for the ± and ∓ signs in the formulas — especially for cos and tan. If you use the sign on the top on the RHS, you have to use the sign on the top on the left-hand side too — so cos(A + B) = cos A cos B – sin A sin B.

Use the **Formulas** to find the **Exact Value** of trig expressions

You should know the value of sin, cos and tan for **common angles** (in **degrees** and **radians**). These values come from using **Pythagoras** on **right-angled triangles —** see p.56.

	0°	30°	45°	60°	90°
	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	n/a

In the exam you might be asked to calculate the **exact value** of sin, cos or tan for **another angle** using your knowledge of those angles and the **addition formulas**.

Find a **pair of angles** from the table which **add or subtract** to give the angle you're after. Then plug them into the **addition formula**, and work it through.

Example: Using the addition formula for tan, show that $\tan 15° = 2 - \sqrt{3}$.

1) Pick two angles that **add or subtract to give 15°**, and put them into the tan addition formula. It's easiest to use **tan 60°** and **tan 45°** here, since neither of them are fractions.

$$\tan 15° = \tan(60° - 45°) = \frac{\tan 60° - \tan 45°}{1 + \tan 60° \tan 45°}$$ ← Using $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

2) **Substitute** the values for tan 60° and tan 45° into the equation: $= \frac{\sqrt{3} - 1}{1 + (\sqrt{3} \times 1)} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$

3) Now rationalise the denominator of the fraction to get rid of the $\sqrt{3}$:

$$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1} = \frac{3 - 2\sqrt{3} + 1}{3 - \sqrt{3} + \sqrt{3} - 1}$$

If you can't remember how to rationalise the denominator have a peek at page 7.

4) Simplify the expression... $= \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$ ...and there's the **right-hand side**.

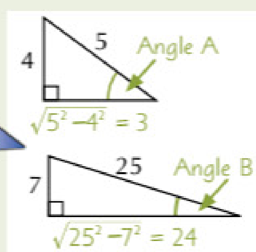
Example: Using the addition formula for sin, find $\sin(A + B)$, where $\sin A = \frac{4}{5}$ and $\sin B = \frac{7}{25}$.

1) Look at the **addition formula** for sin: $\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$

2) You're given sin A and sin B, but you need cos A and cos B too. The numbers in the fractions should make you think of **right-angled triangles**:

The triangles which have $\sin A = \frac{4}{5}$ and $\sin B = \frac{7}{25}$ have $\cos A = \frac{3}{5}$ and $\cos B = \frac{24}{25}$.

Because $\sin = \frac{\text{opp}}{\text{hyp}}$ and $\cos = \frac{\text{adj}}{\text{hyp}}$.



3) Putting these values into the formula gives:

$$\sin(A + B) = \left(\frac{4}{5} \times \frac{24}{25}\right) + \left(\frac{3}{5} \times \frac{7}{25}\right) = \frac{117}{125}$$

Addition and Double Angle Formulas

*There's a **Double Angle Formula** for **Each Trig Function***

Whenever you see a trig expression with an **even multiple of x** in it, like $\sin 2x$, you can use one of the **double angle formulas** to prune it back to an expression just in terms of a **single x**. They are just a slightly different kind of **identity**. You need to know the double angle formulas for **sin**, **cos** and **tan**:

$$\sin 2A \equiv 2\sin A\cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A$$
or $\quad \equiv 2\cos^2 A - 1$
or $\quad \equiv 1 - 2\sin^2 A$

$$\tan 2A \equiv \frac{2\tan A}{1-\tan^2 A}$$

You get these formulas by writing 2A as A + A and using the addition formulas from the previous page.

You can use the identity $\cos^2 A + \sin^2 A \equiv 1$ to get the other versions of the cos 2A formula.

*Use the **Double Angle Formulas** to **Simplify** and **Solve Equations***

If an equation has a **mixture** of **$\sin x$** and **$\sin 2x$** terms in it, there's not much that you can do with it. So that you can **simplify** it, and then **solve** it, you have to use one of the **double angle formulas**.

Example: Solve the equation $\cos 2x - 5\cos x = 2$ in the interval $0 \le x \le 2\pi$.

1) First use the double angle formula $\cos 2A \equiv 2\cos^2 A - 1$ to get rid of $\cos 2x$: $2\cos^2 x - 1 - 5\cos x = 2$

Use this version so that you don't end up with a mix of sin and cos terms.

2) Simplify so you have zero on one side... $\quad 2\cos^2 x - 5\cos x - 3 = 0$

...then factorise and solve the quadratic that you've made: $\quad (2\cos x + 1)(\cos x - 3) = 0$

So $(2\cos x + 1) = 0$ or $(\cos x - 3) = 0$

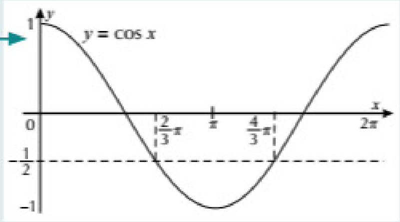
3) The second bracket gives you $\cos x = 3$, which has no solutions since $-1 \le \cos x \le 1$.

4) So all that's left is to solve the first bracket to find x:

$2\cos x + 1 = 0$

$\cos x = -\frac{1}{2} \Rightarrow x = \frac{2}{3}\pi$ or $x = \frac{4}{3}\pi$

Sketch the graph of cos x to find all values of x in the given interval: $\cos x = -\frac{1}{2}$ twice, once at $\frac{2}{3}\pi$ and once at $2\pi - \frac{2}{3}\pi = \frac{4}{3}\pi$.



(Or you can use the CAST method if you prefer — see p.62.)

Practice Questions

Q1 Using the addition formula for cos, find the exact value of $\cos\frac{\pi}{12}$.

Q2 Use the double angle formula to solve the equation: $\sin 2\theta = -\sqrt{3}\sin\theta$, $0° \le \theta \le 360°$.

Q3 Use the double angle formulas to write: a) $\sin\frac{x}{2}\cos\frac{x}{2}$ in terms of $\sin x$, b) $\tan 6x$ in terms of $\tan 3x$.

Exam Questions

Q1 Using double angle and addition identities, find an expression for $\sin 3x$ in terms of $\sin x$ only. [4 marks]

Q2 Using the cos addition formula, show that if $\sin\theta = \cos\left(\frac{\pi}{4} - \theta\right)$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $\theta = 1.18$ to 2 d.p. [4 marks]

Q3 Solve the equation $2\tan 4x = \tan 2x$ for $-90° \le x \le 90°$. [6 marks]

Double the angles, double the fun...

You need to know the double angle formulas off by heart — unlike the addition ones, they WON'T be on the formula sheet. Watch out for questions where you need to use a double angle formula to turn a '4x' into a '2x', four x-ample.

Trigonometric Proofs

They say the proof of the pudding is in the eating, but trigonometry ain't no pudding, treacle.

*Use the **Trig Identities** to prove something is the **Same** as something else*

As well as simplifying and solving nasty trig equations, you can also use **identities** to **prove** or '**show that**' two **trig expressions** are **the same**. A bit like this, in fact:

Example: Show that $\frac{\cos^2\theta}{1+\sin\theta} \equiv 1 - \sin\theta$.

1) Prove things like this by playing about with one side of the equation until you get the other side.

Left-hand side: $\frac{\cos^2\theta}{1+\sin\theta}$

See page 59.

2) The only thing I can think of doing here is replacing $\cos^2\theta$ with $1 - \sin^2\theta$.
(Which is good because it works.)

$$\equiv \frac{1-\sin^2\theta}{1+\sin\theta}$$

The next trick is the hardest to spot. Look at the top — does that remind you of anything?

3) The top line is a **difference of two squares**:

$1 - a^2 = (1 + a)(1 - a)$
$\Rightarrow 1 - \sin^2\theta = (1 + \sin\theta)(1 - \sin\theta)$

$$\equiv \frac{(1+\sin\theta)(1-\sin\theta)}{1+\sin\theta}$$

$\equiv 1 - \sin\theta$, **the right-hand side.**

Example: Show that $\frac{\tan^2 x}{\sec x} \equiv \sec x - \cos x$.

This example uses an identity from page 68.

Again, take one side of the identity and play about with it until you get the other side:

Left-hand side: $\frac{\tan^2 x}{\sec x}$

Try replacing $\tan^2 x$ with $\sec^2 x - 1$: $\equiv \frac{\sec^2 x - 1}{\sec x} \equiv \frac{\sec^2 x}{\sec x} - \frac{1}{\sec x} \equiv \sec x - \cos x$ ...which is the **right-hand side**.

*You might have to use **Addition** or **Double Angle Identities***

You might be asked to use the **addition formulas** to prove an identity (see p.70). Just put the **numbers** and **variables** from the **left-hand side** into the addition formulas and **simplify** until you get the expression you're after.

Example: Prove that $\cos(a + 60°) + \sin(a + 30°) \equiv \cos a$.

Be careful with the + and – signs here.

Put the numbers from the question into the **addition formulas**:

$\cos(a + 60°) + \sin(a + 30°) \equiv (\cos a \cos 60° - \sin a \sin 60°) + (\sin a \cos 30° + \cos a \sin 30°)$

Now **substitute** in any sin and cos values that you know... $= \frac{1}{2}\cos a - \frac{\sqrt{3}}{2}\sin a + \frac{\sqrt{3}}{2}\sin a + \frac{1}{2}\cos a$

See p.56.

...and **simplify**: $= \frac{1}{2}\cos a + \frac{1}{2}\cos a = \cos a$

Maths criminals: innocent until proven sin y.

Whenever you have an expression that contains any angle that's **twice the size** of another, you can use the **double angle formulas** (see p.71)— whether it's $\sin x$ and $\sin 2x$, $\cos 2x$ and $\cos 4x$ or $\tan x$ and $\tan \frac{x}{2}$.

Example: Prove that $2\left(\cot\frac{x}{2}\right)\left(1-\cos^2\frac{x}{2}\right) \equiv \sin x$.

This hellish example uses loads of different identities — there are more like this on the next page too. Woohoo.

First, use the identity $\sin^2\theta + \cos^2\theta \equiv 1$ to replace $1 - \cos^2\frac{x}{2}$ on the left-hand side: **Left-hand side:** $2\cot\frac{x}{2}\sin^2\frac{x}{2}$

Now write $\cot\theta$ as $\frac{\cos\theta}{\sin\theta}$: $2\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\sin^2\frac{x}{2} \equiv 2\cos\frac{x}{2}\sin\frac{x}{2}$

Now you can use the sin 2A double angle formula to write $\sin x \equiv 2\sin\frac{x}{2}\cos\frac{x}{2}$ (using $A = \frac{x}{2}$).

So using the **sin double angle** formula... $\equiv \sin x$ ...you get the **right-hand side**.

Trigonometric Proofs

Small Angle Approximations can pop up too

Remember those lovely **small angle approximations** from p.69? No?
Well you'd better flick back a few pages for a recap before this example.

Example: Show that $\frac{2x\sin 2x}{1-\cos 5x} \approx \frac{8}{25}$ when x is small.

This is a whopping clue (pun intended) as to how you're going to tackle this question. If an exam question mentions an angle being 'small', think 'small angle approximations'.

Use the **small angle approximations** for each trig function, then **simplify**:

$\sin\theta \approx \theta$ $\qquad \cos\theta \approx 1 - \frac{1}{2}\theta^2$

$$\frac{2x\sin 2x}{1-\cos 5x} \approx \frac{2x(2x)}{1-\left(1-\frac{1}{2}(5x)^2\right)} = \frac{4x^2}{\frac{25}{2}x^2} = \frac{8}{25}$$

You might have to use Different Bits of Trig in the Same Question

Some exam questions might try to catch you out by making you use **more than one** identity...

Example: Show that $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$.

You have to use both the addition formula and the double angle formulas in this question.

First, write **$\cos 3\theta$** as **$\cos(2\theta + \theta)$**, then you can use the **cos addition formula**:

$$\cos(3\theta) \equiv \cos(2\theta + \theta) \equiv \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$$

Now you can use the cos and sin **double angle formulas** to get rid of the 2θ:

$$\cos 2\theta\cos\theta - \sin 2\theta\sin\theta \equiv (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$$
$$\equiv 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta \equiv 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta$$
$$\equiv 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta \equiv \mathbf{4\cos^3\theta - 3\cos\theta}$$

This uses the identity $\sin^2\theta + \cos^2\theta \equiv 1$ in the form $\sin^2\theta \equiv 1 - \cos^2\theta$.

This next question looks short and sweet, but it's actually pretty nasty
— you need to know a sneaky **conversion** between **sin** and **cos**.

Example: If $y = \arcsin x$ for $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, show that $\arccos x = \frac{\pi}{2} - y$.

$y = \arcsin x$, so $x = \sin y$ (as arcsin is the inverse of sin — see p.66).

Now the next bit isn't obvious — you need to use an identity to **switch from sin to cos**. This gives: $\quad x = \cos\left(\frac{\pi}{2} - y\right)$

Now, **taking inverses** gives: $\quad \arccos x = \arccos\left[\cos\left(\frac{\pi}{2} - y\right)\right]$

$$\Rightarrow \arccos x = \frac{\pi}{2} - y$$

Converting sin to cos (and back):
$\sin t \equiv \cos\left(\frac{\pi}{2} - t\right)$
and $\cos t \equiv \sin\left(\frac{\pi}{2} - t\right)$.
Remember sin is just cos shifted by $\frac{\pi}{2}$ and vice versa.

Practice Questions

Q1 Show that $\frac{\sin^4 x + \sin^2 x\cos^2 x}{\cos^2 x - 1} \equiv -1$.

Q2 Use trig identities to show that: a) $\cot^2\theta + \sin^2\theta \equiv \operatorname{cosec}^2\theta - \cos^2\theta$, b) $\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} \equiv 2\operatorname{cosec} 2\theta$.

Exam Questions

Q1 a) Show that $\frac{2\sin x}{1-\cos x} - \frac{2\cos x}{\sin x} \equiv 2\operatorname{cosec} x$. [4 marks]

b) Use this result to find all the solutions for which $\frac{2\sin x}{1-\cos x} - \frac{2\cos x}{\sin x} = 4$, $0 < x < 2\pi$. [3 marks]

Q2 Find an approximation for $(4x)^{-1}\operatorname{cosec} 3x\,(2\cos 7x - 2)$, for sufficiently small values of x. [4 marks]

Q3 Prove the identity $\cos\theta\cos 2\theta + \sin\theta\sin 2\theta \equiv \cos\theta$. [4 marks]

Did someone mention treacle pudding...?

And that's your lot for this section. There is nothing left to prove. Except for your well-honed trig skills in the exam...