Vectors

You might have seen vectors before at GCSE. If you haven't, no worries, you're in for a treat. We're going to start with the basics — like what vectors are and how to add them together.

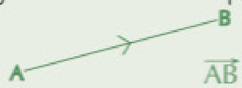
Vectors have Magnitude and Direction — Scalars Don't

1) Vectors have both **size and direction** — e.g. a velocity of 2 m/s on a bearing of 050°, or a displacement of 3 m north. **Scalars** are just quantities **without a direction**, e.g. a speed of 2 m/s, a distance of 3 m.
2) Vectors are drawn as lines with arrowheads on them.
   - The **length** of the line represents the **magnitude** (size) of the vector (e.g. the speed component of velocity). Sometimes vectors are drawn **to scale**.
   - The **direction** of the arrowhead shows the **direction** of the vector.

There are two ways of writing vectors:
1) Using a lower case, bold letter.

When you're handwriting a vector like this, you should underline the letter, i.e. a̲.

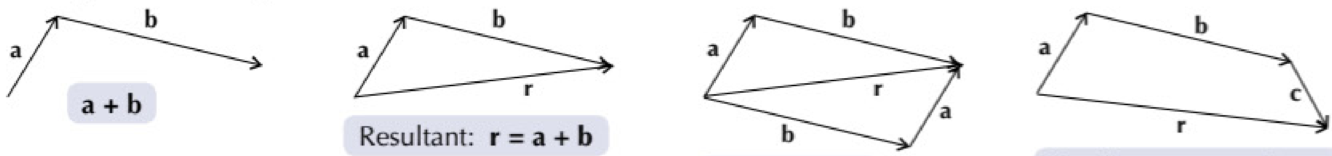
2) Putting an arrow over the endpoints.



Two vectors that have the **same** magnitude and direction are **equal**.

Find the Resultant by Drawing Vectors Nose to Tail

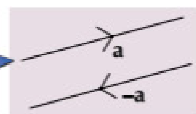
You can add vectors together by drawing the arrows **nose to tail**.
The single vector that goes from the start to the end of the vectors is called the **resultant** vector.



a + b   Resultant: **r = a + b**   **a + b = b + a**   Resultant: **r = a + b + c**

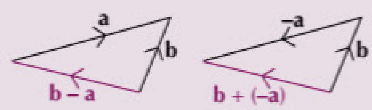
Subtracting a Vector is the Same as Adding a Negative Vector

1) The vector **–a** is in the **opposite direction** to the vector **a**. They're both the **same size**.



2) So **subtracting a vector** is the same as **adding the negative vector**:

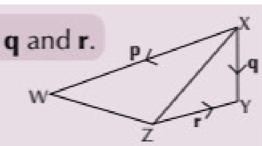
b – a = b + (–a)



3) You can use the adding and subtracting rules to find a vector **in terms of other vectors**.

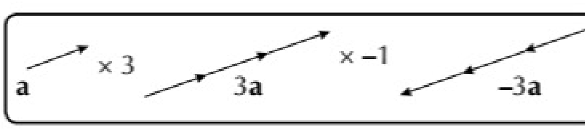
Example: Find $\overrightarrow{WZ}$ and $\overrightarrow{ZX}$ in terms of **p**, **q** and **r**.

$\overrightarrow{WZ}$ = **–p + q – r** $\overrightarrow{ZX}$ = **r – q**



Vectors a, 2a and 3a are all Parallel

You can **multiply** a vector by a **scalar** (just a number, remember) — the **length changes** but the **direction stays the same**.



Multiplying a vector by a non-zero scalar always produces a **parallel vector**.

All these vectors are **parallel**: **9a + 15b**   **–18a – 30b**   **6a + 10b**

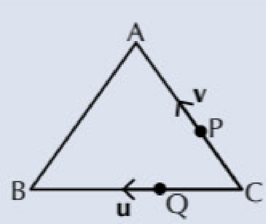
This is $\frac{2}{3}$(9a + 15b).

This is –2(9a + 15b).

To show that two vectors are **parallel**, you just need to show they are **scalar multiples** of each other.

Example: $\overrightarrow{CA}$ = **v**, $\overrightarrow{CB}$ = **u**. P divides $\overrightarrow{CA}$ in the ratio 1:2, Q divides $\overrightarrow{CB}$ in the ratio 1:2. Show that $\overrightarrow{PQ}$ is parallel to $\overrightarrow{AB}$.

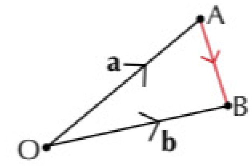
$\overrightarrow{AB}$ = **–v + u**. P divides $\overrightarrow{CA}$ in the ratio 1:2 so P is one third of the way along $\overrightarrow{CA}$.
This means $\overrightarrow{CP} = \frac{1}{3}\mathbf{v}$, so $\overrightarrow{PC} = -\frac{1}{3}\mathbf{v}$. Similarly, $\overrightarrow{CQ} = \frac{1}{3}\mathbf{u}$.
So, $\overrightarrow{PQ} = -\frac{1}{3}\mathbf{v} + \frac{1}{3}\mathbf{u} = \frac{1}{3}(-\mathbf{v} + \mathbf{u}) = \frac{1}{3}\overrightarrow{AB}$. This shows that $\overrightarrow{PQ}$ is parallel to $\overrightarrow{AB}$.



Vectors

Position Vectors Describe Where a Point Lies

You can use a vector to describe the **position of a point**, in relation to the **origin, O**.

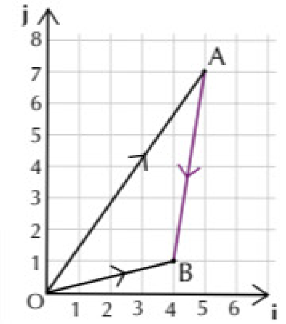


The position vector of point A is $\overrightarrow{OA}$. It's usually called vector **a**.
The position vector of point B is $\overrightarrow{OB}$. It's usually called vector **b**.

You can write other vectors in terms of position vectors: $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$

Vectors can be described using i and j Units or Column Vectors

1) A **unit vector** is any vector with a **magnitude of 1 unit**.
2) The vectors **i** and **j** are **standard unit vectors**. **i** is in the direction of the *x*-**axis**, and **j** is in the direction of the *y*-**axis**. They each have a magnitude of **1 unit**.
3) You use them to say how far **horizontally** and **vertically** you have to go to get from the start of the vector to the end.



The position vector of point A = $\mathbf{a} = 5\mathbf{i} + 7\mathbf{j}$
The position vector of point B = $\mathbf{b} = 4\mathbf{i} + \mathbf{j}$

This tells you that point B lies 4 units to the right and 1 unit above the origin — it's just like coordinates.

Vector $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
$= (4\mathbf{i} + \mathbf{j}) - (5\mathbf{i} + 7\mathbf{j})$
$= -\mathbf{i} - 6\mathbf{j}$

Add/subtract the **i** and **j** components separately.

To go from A to B, you go 1 unit left and 6 units down. It's just like a translation.

4) Column vectors are a really easy way of writing out vectors. $x\mathbf{i} + y\mathbf{j} = \begin{pmatrix} x \\ y \end{pmatrix}$
5) **Calculating** with them is a breeze. Just add or subtract the **top row**, then add or subtract the **bottom row** separately.

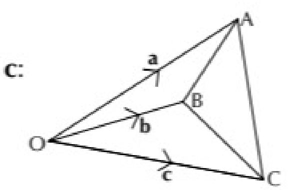
$\mathbf{a} = 5\mathbf{i} + 7\mathbf{j} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$ $\quad \mathbf{b} = 4\mathbf{i} + \mathbf{j} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ -6 \end{pmatrix}$

6) When you're **multiplying** a column vector by a **scalar**, you multiply **each number** in the column vector by the scalar.

$2\mathbf{b} - 3\mathbf{a} = 2\begin{pmatrix} 4 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix} - \begin{pmatrix} 15 \\ 21 \end{pmatrix} = \begin{pmatrix} -7 \\ -19 \end{pmatrix}$

Practice Questions



Q1 Using the diagram on the right, find these vectors in terms of vectors **a**, **b** and **c**:
a) $\overrightarrow{AB}$ b) $\overrightarrow{BA}$ c) $\overrightarrow{CB}$ d) $\overrightarrow{AC}$

Q2 Give the position vector of point P, which has the coordinates (2, –4). Give your answer in unit vector form.

Q3 Given that vectors **a** and **b** have position vectors $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$ respectively, what is $3\mathbf{a} - 2\mathbf{b}$?

Exam Questions

Q1 The points W, X and Y have position vectors $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ respectively.
Find the position vector of Z such that $\overrightarrow{WX} = \overrightarrow{YZ}$. [3 marks]

Q2 The points A and B have position vectors $-2\mathbf{i} + 4\mathbf{j}$ and $5\mathbf{i} + \mathbf{j}$ respectively.
Point P is on the line AB such that AP : PB = 1 : 3. Determine the position vector of P. [4 marks]

I've got B and Q units in my kitchen...

If you're asked to show that two vectors are parallel, remember that this is the same as showing they are scalar multiples of each other. So, just show that you can write one as the other multiplied by a scalar and you're laughing.

More Vectors

The magnitude of a vector is its length. The direction of a vector is the angle that the vector makes with the horizontal axis. I know that you're oh so eager to know how to calculate these, so without further ado...

Use **Pythagoras' Theorem** to Find Vector **Magnitudes**

A vector's magnitude is sometimes called its modulus.

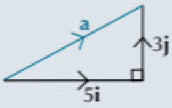
1) The **magnitude** of vector **a** is written as $|\mathbf{a}|$, and the magnitude of $\overrightarrow{AB}$ is written as $|\overrightarrow{AB}|$.
2) The **i** and **j** components of a vector form a **right-angled triangle**, so you can use the **Pythagoras formula** to find a vector's magnitude.
3) You might be asked to find a **unit vector** in the direction of a particular vector.

A unit vector in the direction of vector $\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$

Remember — a unit vector has a magnitude of 1 (see the previous page).

Example: Find the unit vector in the direction of $\mathbf{a} = 5\mathbf{i} + 3\mathbf{j}$.

First find the magnitude of **a**:



$|\mathbf{a}| = \sqrt{5^2 + 3^2} = \sqrt{34} = 5.83...$

So, the unit vector is:

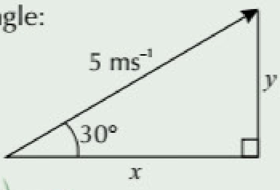
$$\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}}{\sqrt{34}} = \frac{1}{\sqrt{34}}(5\mathbf{i} + 3\mathbf{j})$$

Resolving Means Writing a Vector as **Component Vectors**

1) Splitting a vector up into **i** and **j** vectors means you can work things out with **one component at a time**. When **adding** vectors to get a **resultant vector**, it's easier to **add** the **horizontal** and **vertical** components **separately**.
2) So you **split** the vector into components first — this is called **resolving the vector**.
3) You use a combination of **trig** and **Pythagoras' theorem** to **resolve** a vector into its **component form**.

Example: A ball is travelling at 5 ms^{-1} at an angle of 30° to the horizontal. Find the horizontal and vertical components of the ball's velocity, $\boldsymbol{v}$.

First, draw a diagram and make a right-angled triangle:



Use trigonometry to find x and y:

$\cos 30° = \frac{x}{5} \Rightarrow x = 5 \cos 30°$

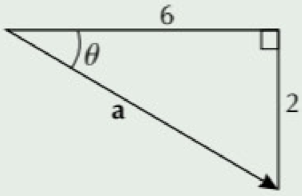
$\sin 30° = \frac{y}{5} \Rightarrow y = 5 \sin 30°$

So $\boldsymbol{v} = (5 \cos 30°\mathbf{i} + 5 \sin 30°\mathbf{j})\ \text{ms}^{-1} = \left(\frac{5\sqrt{3}}{2}\mathbf{i} + \frac{5}{2}\mathbf{j}\right)\ \text{ms}^{-1}$

Victor and Hector decided to resolve this the old-fashioned way...

4) The **direction** of a vector is usually measured going **anticlockwise** from the **positive x-axis**. Give the direction in this form unless the question implies otherwise (e.g. if the direction is a bearing).

Example: The acceleration of a body is given by the vector $\mathbf{a} = 6\mathbf{i} - 2\mathbf{j}\ \text{ms}^{-2}$. Find the magnitude and direction of the acceleration.



Start with a diagram again. Remember, the y-component "–2" means "down 2".

Using Pythagoras' theorem, you can work out the magnitude of **a**:

$|\mathbf{a}|^2 = 6^2 + (-2)^2 = 40 \Rightarrow |\mathbf{a}| = \sqrt{40} = 6.32\ \text{ms}^{-2}$ (3 s.f.)

Use trigonometry to work out the angle:

$\tan\theta = \frac{2}{6} \Rightarrow \theta = \tan^{-1}\left(\frac{2}{6}\right) = 18.4°$ (1 d.p.)

So the direction of **a** is 360° – 18.4° = **341.6°** (1 d.p.)

So vector **a** has magnitude **6.32 ms^{-2}** (3 s.f.) and direction **341.6°** (1 d.p.).

The angle $\theta = \tan^{-1}\frac{y}{x}$ and the direction are often different — drawing a diagram can help you figure out what's what.

In general, a vector with magnitude r and direction θ can be written as $r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$.

The vector $x\mathbf{i} + y\mathbf{j}$ has magnitude $r = \sqrt{x^2 + y^2}$ and makes an angle of $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ with the horizontal.

More Vectors

Use a *Vector's Magnitude* to Find the *Distance Between Two Points*

You can calculate the **distance between two points** by finding the **vector** between them, and then using **Pythagoras' Theorem** to calculate its **magnitude**.

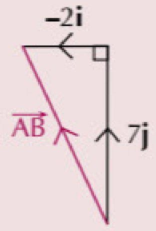
Example: The position vectors of points A and B are $4\mathbf{i} - 2\mathbf{j}$ and $2\mathbf{i} + 5\mathbf{j}$ respectively. Calculate the distance between points A and B to 2 decimal places.

Find the vector $\overrightarrow{AB}$ between the points.

$\overrightarrow{AB} = 2\mathbf{i} + 5\mathbf{j} - (4\mathbf{i} - 2\mathbf{j}) = (2 - 4)\mathbf{i} + (5 - (-2))\mathbf{j} = -2\mathbf{i} + 7\mathbf{j}$.

Draw a diagram, and then calculate the magnitude of $\overrightarrow{AB}$ using Pythagoras' theorem.

$|\overrightarrow{AB}| = \sqrt{(-2)^2 + 7^2} = \sqrt{53}$ so the distance between points A and B is **7.28** (2 d.p.).



Use the *Cosine Rule* to Find the *Angle Between Two Vectors*

The angle between two vectors **a** and **b** can be calculated by constructing a triangle with **a** and **b** as two of its sides. First, calculate the **magnitude** of these vectors, then use the **cosine rule** (see p.58) to find the angle between them.

Example: Find the angle θ between the vectors $\overrightarrow{PQ} = 3\mathbf{i} - \mathbf{j}$ and $\overrightarrow{PR} = -\mathbf{i} + 4\mathbf{j}$.

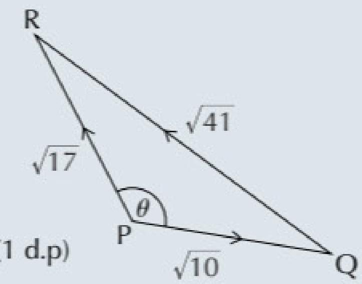
The side lengths of the triangle PQR are:

$\overrightarrow{QR} = \overrightarrow{PR} - \overrightarrow{PQ} = -\mathbf{i} + 4\mathbf{j} - (3\mathbf{i} - \mathbf{j}) = -4\mathbf{i} + 5\mathbf{j}$

$|\overrightarrow{PQ}| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$, $|\overrightarrow{PR}| = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$ and $|\overrightarrow{QR}| = \sqrt{(-4)^2 + 5^2} = \sqrt{41}$

Use the cosine rule to find angle θ:

$$\cos\theta = \frac{(\sqrt{10})^2 + (\sqrt{17})^2 - (\sqrt{41})^2}{2 \times \sqrt{17} \times \sqrt{10}} = \frac{-14}{2\sqrt{170}} = \frac{-7}{\sqrt{170}}, \text{ so: } \theta = \cos^{-1}\left(\frac{-7}{\sqrt{170}}\right) = \mathbf{122.5°} \text{ (1 d.p)}$$



Practice Questions

Q1 Find the unit vector in the direction of $\mathbf{q} = -2\mathbf{i} + 5\mathbf{j}$.

Q2 If A = (1, 2) and B = (3, –1), find: a) $|\overrightarrow{OA}|$ b) $|\overrightarrow{OB}|$ c) $|\overrightarrow{AB}|$

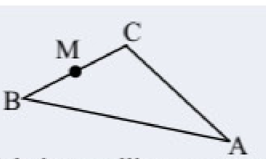
Q3 A vector **s** has a magnitude of 7 and direction 20° above the horizontal. Write **s** in the form $a\mathbf{i} + b\mathbf{j}$.

Q4 The velocity of a ball is modelled with the vector $\mathbf{v} = -\mathbf{i} + 3\mathbf{j}$. Calculate the direction of the vector **v**.

Q5 The position vector of point X is $\begin{pmatrix} 7 \\ -1 \end{pmatrix}$, the position vector of point Y is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Calculate the distance between points X and Y. Hence find the angle between vectors $\overrightarrow{OX}$ and $\overrightarrow{OY}$.

Exam Questions

Q1 The following sketch shows a triangle ABC.
M is the midpoint of line BC.
Given that $\overrightarrow{AB} = -5\mathbf{i} + 2\mathbf{j}$ and $\overrightarrow{AC} = -2\mathbf{i} + 4\mathbf{j}$, find $|\overrightarrow{AM}|$. [3 marks]



Q2 a) The movement of a particle is modelled by vector **p**. The particle is travelling at a speed of 7 m/s with direction of 15° above the horizontal. Write **p** in component form. [2 marks]

b) The particle strikes another particle. Its movement is now modelled by vector $\mathbf{q} = 2\sqrt{2}(\mathbf{i} + \mathbf{j})$. Find the amount by which the particle's speed has decreased and state the particle's new direction. [3 marks]

Q3 Points S, T, U and V make a parallelogram, STUV. The position vectors of S, T and V are $\mathbf{i} + \mathbf{j}$, $8\mathbf{i} + \mathbf{j}$ and $-\mathbf{i} + 5\mathbf{j}$ respectively. Find the distance between S and U. Leave your answer in exact surd form. [4 marks]

I don't think you've quite grasped the magnitude of the situation...

The magnitude is just a scalar, which means it doesn't have a direction — i.e. $|\overrightarrow{AB}| = |\overrightarrow{BA}|$. Squaring the numbers gets rid of any minus signs, so it doesn't make any difference which way round you subtract the coordinates. Superb.