The **Best** has been saved till last...

These two identities are really important. You'll need them **loads**.

$$\tan x \equiv \frac{\sin x}{\cos x}$$

$$\sin^2 x + \cos^2 x \equiv 1$$

$$\Rightarrow \sin^2 x \equiv 1 - \cos^2 x$$
$$\cos^2 x \equiv 1 - \sin^2 x$$

Work out these two using $\sin^2 x + \cos^2 x \equiv 1$.

Essential kit for maintaining two identities.

These two come up in exam questions **all the time**. Learn them. Learnthemlearnthemlearnthemlearnthemlearnthemlear... okay, I'll stop now.

Practice Questions

Q1 Find the missing sides and angles in: a) ΔABC, in which $A = 30°$, $C = 25°$, $b = 6$ m, and find its area.
b) ΔPQR, in which $p = 3$ km, $q = 23$ km, $R = 10°$. (answers to 2 d.p.)

Q2 My pet triangle Freda has sides of length 10, 20 and 25. Find her angles (in degrees to 1 d.p.).

Exam Questions

Q1 For an angle x, $3 \cos x = 2 \sin x$. Find $\tan x$. [2 marks]

Q2 Two walkers, X and Y, walked in different directions from the same start position.
X walked due south for 150 m. Y walked 250 m on a bearing of 100°.

a) Calculate the final distance between the two walkers, in m to the nearest m. [2 marks]

b) θ is the final bearing of Y from X. Show that $\frac{\sin \theta}{\sin 80°} = 0.93$ to 2 decimal places. [3 marks]

Tri angles — go on... you might like them.

Formulas and trigonometry go together even better than Ant and Dec. I can count 7 formulas on these pages. That's not many, so please, make sure you know them. If you haven't learnt them I will cry for you. I will sob.

Using Trig Identities

Trig Identities are useful for *Solving Equations*...

$$\tan x \equiv \frac{\sin x}{\cos x}$$

$$\sin^2 x + \cos^2 x \equiv 1 \Rightarrow \begin{matrix} \sin^2 x \equiv 1 - \cos^2 x \\ \cos^2 x \equiv 1 - \sin^2 x \end{matrix}$$

If you've got a trig equation with a **tan** in it, together with a sin or a cos — chances are you'll be better off if you rewrite the tan using the first identity above. The other identities are useful for getting rid of $\mathbf{sin^2}\ x$ and/or $\mathbf{cos^2}\ x$ (see next page).

Example: Solve: $3 \sin x - \tan x = 0$, for $0 \le x \le 360°$.

1) It's got **sin** and **tan** in it — so writing $\tan x$ as $\frac{\sin x}{\cos x}$ is probably a good move:

$$3 \sin x - \tan x = 0$$

$$\Rightarrow 3 \sin x - \frac{\sin x}{\cos x} = 0$$

2) Get rid of the **cos *x*** on the bottom by multiplying the whole equation by $\cos x$.

$$\Rightarrow 3 \sin x \cos x - \sin x = 0$$

3) Now — there's a **common factor** of $\sin x$. Take that outside a bracket.

$$\Rightarrow \sin x\,(3 \cos x - 1) = 0$$

4) And now you're almost there. You've got two things multiplying together to make zero. That means either **one or both** of them is **equal to zero**.

$$\Rightarrow \sin x = 0 \text{ or } 3 \cos x - 1 = 0$$

CAST gives any solutions in the interval $0 \le x \le 360°$.

$\sin x = 0$

The first solution is:

$$\sin 0 = 0$$

Now find the other points where $\sin x$ is zero in the interval $0 \le x \le 360°$.

Remember the sin graph is zero every 180°.

$\Rightarrow x = 0°, 180°, 360°$

Having memorised the roots of sin x, smug young Sherlock had ample time to entertain his classmates as they caught up.

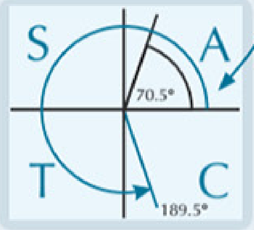
$3 \cos x - 1 = 0$

Rearrange:

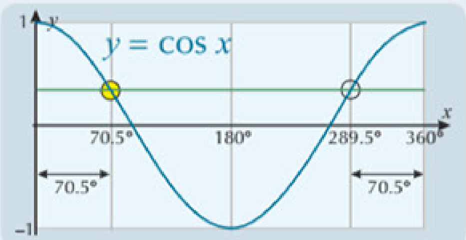
$$\cos x = \frac{1}{3}$$

So the first solution is:

$$\cos^{-1}\left(\frac{1}{3}\right) = 70.52877...° = 70.5° \text{ (1 d.p.)}$$



CAST (or the graph of $\cos x$) gives another positive solution in the 4th quadrant, where $x = 360° - 70.52877...° = 289.5°$ (1 d.p.)



And the two solutions from this part are:

$\Rightarrow x = 70.5°, 289.5°$

So altogether you've got **five** possible solutions:

$\Rightarrow$ **$x = 0°, 70.5°, 180°, 289.5°, 360°$**

Be warned — you might be tempted to simplify an equation by **dividing** by a trig function. But you can **only** do this if the trig function you're dividing by is **never zero** in the range the equation is valid for. Dividing by zero is not big or clever, or even possible.

Using Trig Identities

Use the identity $\sin^2 x + \cos^2 x = 1$ to get rid of $\sin^2 x$ or $\cos^2 x$

Example: Solve: $2\sin^2 x + 5\cos x = 4$, for $0° \le x \le 360°$.

1) You can't do much while the equation's got both sin's and cos's in it. So replace the $\sin^2 x$ bit with $1 - \cos^2 x$:

$$2(1 - \cos^2 x) + 5\cos x = 4$$

Now the only trig function is cos.

2) Multiply out the bracket and rearrange it so that you've got zero on one side — and you get a **quadratic** in $\cos x$:

$$\Rightarrow 2 - 2\cos^2 x + 5\cos x = 4$$
$$\Rightarrow 2\cos^2 x - 5\cos x + 2 = 0$$

3) This is a quadratic in $\cos x$. It's easier to factorise this if you make the substitution $y = \cos x$.

$$2y^2 - 5y + 2 = 0$$
$$\Rightarrow (2y - 1)(y - 2) = 0$$
$$\Rightarrow (2\cos x - 1)(\cos x - 2) = 0$$

$2y^2 - 5y + 2 = (2y \;?)(y \;?) = (2y - 1)(y - 2)$

4) Now one of the brackets must be **0**. So you get 2 equations as usual:

$$2\cos x - 1 = 0 \quad \text{or} \quad \cos x - 2 = 0$$
$$\cos x = \frac{1}{2} \Rightarrow x = 60° \text{ and } x = 300° \quad \text{and} \quad \cos x = 2$$

You did this example on page 38.

This is a bit weird. cos x is always between –1 and 1, so you don't get any solutions from this bracket.

So at the end of all that, the only solutions you get are $x = 60°$ and $x = 300°$. How boring.

Use the Trig Identities to Prove something is the Same as something else

Example: Prove that $\frac{\cos^2\theta}{1 + \sin\theta} \equiv 1 - \sin\theta$.

1) Play about with one side of the equation. **Left-hand side:** $\frac{\cos^2\theta}{1 + \sin\theta}$

2) Replace $\cos^2\theta$ with $1 - \sin^2\theta$. $\equiv \frac{1 - \sin^2\theta}{1 + \sin\theta}$ See page 37.

3) The top line is a **difference of two squares**: $\equiv \frac{(1 + \sin\theta)(1 - \sin\theta)}{1 + \sin\theta}$ ($1 - a^2 = (1 + a)(1 - a)$)

$\equiv 1 - \sin\theta$, **the right-hand side.**

Practice Questions

Q1 Find all the solutions to $6\sin^2 x = \cos x + 5$ in the range $0 \le x \le 360°$ (answers to 1 d.p. where appropriate).

Q2 Solve $3\tan x + 2\cos x = 0$ for $-90° \le x \le 90°$.

Q3 Simplify: $(\sin y + \cos y)^2 + (\cos y - \sin y)^2$.

Q4 Prove that $\frac{\sin^4 x + \sin^2 x\cos^2 x}{\cos^2 x - 1} \equiv -1$.

Exam Questions

Q1 a) Show that the equation $2(1 - \cos x) = 3\sin^2 x$ can be written as $3\cos^2 x - 2\cos x - 1 = 0$. [2 marks]

b) Use this to solve the equation $2(1 - \cos x) = 3\sin^2 x$ for $0 \le x \le 360°$, giving your answers to 1 d.p. [6 marks]

Q2 Solve the equation $3\cos^2 x = \sin^2 x$, for $-180° \le x \le 180°$. [6 marks]

Always do trigonometry on holiday — you'll get a great tan...

You can bet your last penny that you'll need to use a trig identity in the exam. That substitution trick to get rid of a sin² or a cos² and end up with a quadratic in sin x or cos x is a real examiners' favourite. Remember to use CAST or graphs to find all the possible solutions in the given interval, not just the one on your calculator display, and check your answers.

Solving Trig Equations

I used to really hate trig stuff like this. But once I'd got the hang of it, I just couldn't get enough. I stopped going out, lost interest in romance — the CAST method became my life. Learn it, but be careful. It's addictive.

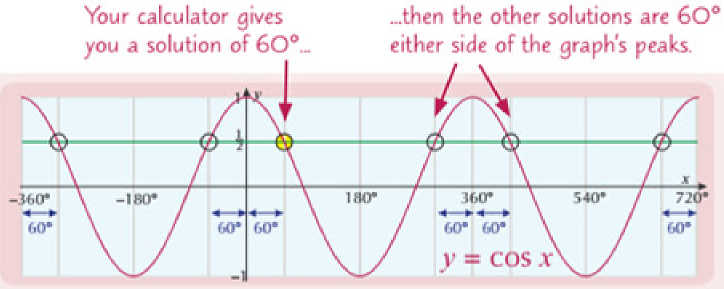
*There are **Two Ways** to find **Solutions** in an **Interval***

Example: Solve $\cos x = \frac{1}{2}$ for $-360° \leq x \leq 720°$.

Like I said — there are **two ways** to solve this kind of question. Just use the one you prefer...

*You can draw a **graph**...*

1) Draw the **graph** of $y = \cos x$ for the range you're interested in...
2) Get the first solution from your **calculator** and mark this on the graph,
3) Use the **symmetry of the graph** to work out what the other solutions are:



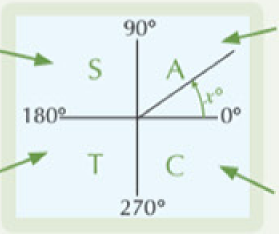
So the solutions are: **–300°**, **–60°**, **60°**, **300°**, **420°** and **660°**.

*...or you can use the **CAST** diagram*

CAST stands for **COS**, **ALL**, **SIN**, **TAN** — and the CAST diagram shows you where these functions are **positive**:

Between 90° and 180°, only **Sin** is positive.

Between 0 and 90°, **All** of sin, cos and tan are positive.



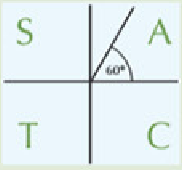
Between 180° and 270°, only **Tan** is positive.

Between 270° and 360°, only **Cos** is positive.

First, to find all the values of x between 0° and 360° where $\cos x = \frac{1}{2}$ — you do this:

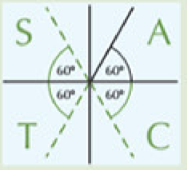
This is positive — so you're only interested in where cos is positive.

Put the first solution onto the CAST diagram.



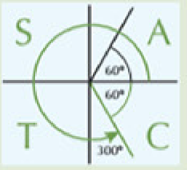
The angle from your calculator goes anticlockwise from the x-axis (unless it's negative — then it would go clockwise into the 4th quadrant).

Find the other angles between 0° and 360° that might be solutions.



The other possible solutions come from making the same angle from the horizontal axis in the other 3 quadrants.

Ditch the ones that are the wrong sign.



cos x = ½, which is positive. The CAST diagram tells you cos is positive in the 4th quadrant — but not the 2nd or 3rd — so ditch those two angles.

So you've got solutions 60° and 300° in the range 0° to 360°. But you need **all the solutions** in the range **–360° to 720°**. Get these by repeatedly **adding or subtracting 360°** onto each until you go out of range:

$x = 60° \Rightarrow$ (adding 360°) $x = 420°$, 780° (too big)

and (subtracting 360°) $x = -300°$, –660° (too small)

$x = 300° \Rightarrow$ (adding 360°) $x = 660°$, 1020° (too big)

and (subtracting 360°) $x = -60°$, –420° (too small)

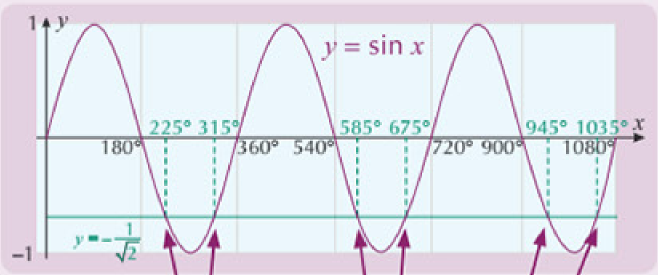
So the solutions are: x = **–300°**, **–60°**, **60°**, **300°**, **420°** and **660°**.

Solving Trig Equations

*Sometimes you end up with **sin kx = number**...*

For these, you might find it easier to draw the **graph** rather than use the CAST method — that's one reason why being able to sketch trig graphs properly is so important.

Example: Solve: $\sin 3x = -\frac{1}{\sqrt{2}}$ for $0° \le x \le 360°$.



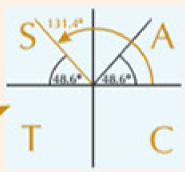
1) You've got $3x$ instead of x, which means the range you need to find solutions in is **$0° \le 3x \le 1080°$**. So draw the graph of $y = \sin x$ between 0° and 1080°.
2) Use your calculator to find the first solution. You'll get **$3x = -45°$**. But this is **outside the range** for $3x$, so use the pattern of the graph to find a solution in the range. As the sin curve repeats every 360°, there'll be a solution at 360 – 45 = **315°**.
3) Now use your graph to find the other 5 solutions. You can see that there's another solution at 180 + 45 = **225°**. Then add on 360° and 720° to both 225° and 315° to get: $3x = 225°, 315°, 585°, 675°, 945°$ and $1035°$.
4) **Divide by 3** to get the solutions for x: $x =$ **75°**, **105°**, **195°**, **225°**, **315°** and **345°**.
5) **Check** your answers by putting these values back into your calculator.

These are the solutions — but remember that this is for 3x. You'll need to divide by 3 to get your final answers.

It really is mega-important that you check these answers — it's dead easy to make a silly mistake. They should all be in the range $0° \le x \le 360°$.

*...or **sin (x + k) = number***

Example: Solve $\sin (x + 60°) = \frac{3}{4}$ for $-360° \le x \le 360°$, giving your answers to 1 d.p.



1) You've got $\sin (x + 60°)$ instead of $\sin x$ — so the range is **$-300° \le x + 60° \le 420°$**.
2) Use your calculator to get the first solution: $x + 60° =$ **48.6°** (1 d.p.)
3) Use a CAST diagram to get the second solution: 180° – 48.6° = **131.4°**
4) Add and subtract 360° to the first two solutions to find the rest:
   48.6° + 360° = 408.6°, 48.6° – 360° = –311.4°, 131.4° + 360° = 491.4°, 131.4° – 360° = –228.6°
   So, for $(x + 60°)$ the solutions to 1 d.p. are: $x + 60° = -228.6°, 48.6°, 131.4°, 408.6°$
5) Subtract 60° to get the solutions for x: $x =$ **–288.6°**, **–11.4°**, **71.4°** and **348.6°**

The red answers are outside $-300° \le x + 60° \le 420°$.

Always check your answers.

Practice Question

Q1 a) Solve each of these equations for $0 \le \theta \le 360°$: (i) $\sin \theta = -\frac{\sqrt{3}}{2}$, (ii) $\tan \theta = -1$, (iii) $\cos \theta = -\frac{1}{\sqrt{2}}$

b) Solve each of these equations for $-180° \le \theta \le 180°$ (giving your answer to 1 d.p.):
(i) $\cos 4\theta = -\frac{2}{3}$ (ii) $\sin (\theta + 35°) = 0.3$ (iii) $\tan \left(\frac{1}{2}\theta\right) = 500$

Exam Question

Q1 a) Solve $2 \cos (x - 45°) = \sqrt{3}$, for $0 \le x \le 360°$. [3 marks]

b) Solve $\sin 2x = -\frac{1}{2}$, for $0° \le x \le 360°$. [3 marks]

Trig equations are sinful (and cosful and tanful)...

Finding all the correct solutions can be tricky so take your time and make sure you check those answers for Pete's sake — or for your own sake if not for old Pete. You can use a trig graph or CAST diagram to find other solutions — just use whichever method you find most comfortable. With plenty of practice, these questions can be a banker in the exam.