Summary of key points

6 You can use the identity $\sin^2 x + \cos^2 x \equiv 1$ to prove the following identities:

- $1 + \tan^2 x \equiv \sec^2 x$
- $1 + \cot^2 x \equiv \text{cosec}^2 x$

Trigonometric identities 2

You will need to learn these trigonometric identities for your A-level exam. Make sure you can prove them both, using the identities $\sin^2\theta + \cos^2\theta \equiv 1$ and $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$

1 $\sec^2\theta \equiv 1 + \tan^2\theta$

$$1 \equiv \sin^2\theta + \cos^2\theta$$
$$\frac{1}{\cos^2\theta} \equiv \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta}$$
$$\sec^2\theta \equiv \tan^2\theta + 1$$
$$\equiv 1 + \tan^2\theta$$

2 $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$

$$1 \equiv \sin^2\theta + \cos^2\theta$$
$$\frac{1}{\sin^2\theta} \equiv \frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta}$$
$$\text{cosec}^2\theta \equiv 1 + \frac{1}{\tan^2\theta}$$
$$\equiv 1 + \cot^2\theta$$

If you have to prove an identity you should start with one side, then rearrange it using identities you know until it looks like the other side. It's usually safest to start with the **left-hand side**. Use the fact that $\sec^4 x = (\sec^2 x)^2$, then multiply out the brackets and rearrange. You can use identity 1 above a second time to complete the proof.

Worked example

Prove that $\sec^4 x - \tan^4 x \equiv \sec^2 x + \tan^2 x$ **(3 marks)**

$$\sec^4 x - \tan^4 x \equiv (1 + \tan^2 x)^2 - \tan^4 x$$
$$\equiv 1 + 2\tan^2 x + \tan^4 x - \tan^4 x$$
$$\equiv 1 + \tan^2 x + \tan^2 x$$
$$\equiv \sec^2 x + \tan^2 x$$

You could also start by writing $(\sec^4 x - \tan^4 x) \equiv (\sec^2 x - \tan^2 x)(\sec^2 x + \tan^2 x)$, then use identity 1 above to show that $\sec^2 x - \tan^2 x \equiv 1$

Worked example

Solve, for $0 \leqslant x \leqslant 180°$, the equation

$\text{cosec}^2 2x - 3\cot 2x = 1$

Give your answers in degrees to 1 decimal place. **(7 marks)**

$1 + \cot^2 2x - 3\cot 2x = 1$

$\cot^2 2x - 3\cot 2x = 0$

$\cot 2x(\cot 2x - 3) = 0$

$\cot 2x = 0$ ✓ $\cot 2x = 3$ ✓

$\tan 2x \to \infty$ $\tan 2x = \frac{1}{3}$

$2x = 90°, 270°$ $2x = 18.43...°, 198.43...°$

$x = 45°, 135°, 9.2°, 99.2°$ (1 d.p.)

1. Use $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$ with $\theta = 2x$ to get a **quadratic** equation in $\cot 2x$.
2. **Factorise** the left-hand side to find two values of $\cot 2x$.
3. Use $\cot 2x = \dfrac{1}{\tan 2x}$ to find all the possible values of $2x$.
4. Divide by 2 to find the solutions.

Remember to transform the range:
$0 \leqslant x \leqslant 180°$ so $0 \leqslant 2x \leqslant 360°$
You are interested in all the possible values of $2x$ between 0° and 360°.

Now try this

1 Prove that $\text{cosec}^2 x - \sec^2 x \equiv \cot^2 x - \tan^2 x$

Answer **(3 marks)**

Rearrange identity 1 at the top of the page: $\tan^2\theta = \sec^2\theta - 1$

2 Solve, for $0 \leqslant \theta \leqslant 360°$, the equation

$3\tan^2\theta + 7\sec\theta = 3$

Give your answers in degrees to 1 decimal place.

Answer **(6 marks)**

Trigonometric equations 3

You need to be able to solve trig equations involving sec, cosec and cot. Make sure you are confident solving trig equations involving sin, cos and tan before revising this topic.

Multiple solutions

You can use this table to find **two solutions** to a trig equation without sketching a graph.
α is the **principal value** you get from your calculator.

Function	Radians	Degrees
$\sin x = k$	$x = \alpha,\ \pi - \alpha$	$x = \alpha,\ 180° - \alpha$
$\cos x = k$	$x = \alpha,\ -\alpha$	$x = \alpha,\ -\alpha$
$\tan x = k$	$x = \alpha,\ \pi + \alpha$	$x = \alpha,\ 180° + \alpha$

You can find **all other solutions** by adding multiples of 2π radians, or 360° to these values.

Worked example

Solve, for $0 \leqslant x \leqslant \pi$,

$\operatorname{cosec} 3x + 2 = 0$

Give your answers in terms of π. **(5 marks)**

$\operatorname{cosec} 3x = -2$

$\frac{1}{\sin 3x} = -2$ so $\sin 3x = -\frac{1}{2}$

$0 \leqslant x \leqslant \pi$, so $0 \leqslant 3x \leqslant 3\pi$

$3x = \cancel{-\frac{\pi}{6}}, \frac{7\pi}{6}, \frac{11\pi}{6}, \cancel{\frac{19\pi}{6}}, \ldots$

$x = \frac{7\pi}{18}, \frac{11\pi}{18}$

Rearrange the equation so it is in the form sin … = … then solve it in the normal way. Remember to transform the range so you find all the possible solutions.

Check it!
You can use your calculator to check your answers.

$$\frac{1}{\sin\left(3 \times \frac{11\pi}{18}\right)} + 2$$

0

Problem solved!

Look out for equations that need to be **factorised**. This equation is the same as $2A^2 + 7A - 4 = 0$, with $A = \sec x$. When solving quadratic equations involving $\sec x$ or $\operatorname{cosec} x$, both factors won't necessarily give you valid answers. The equations $\sec x = k$ and $\operatorname{cosec} x = k$ have **no solutions** for $-1 < k < 1$, so $\sec x = \frac{1}{2}$ doesn't produce any valid solutions.

The equation $\cot x = k$ does have solutions for any value of k, though. Have a look at the graphs on page 78 to see why this is true.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Worked example

Solve, for $-180° \leqslant x \leqslant 180°$,

$2\sec^2 x + 7\sec x = 4$

Give your answer in degrees to 1 decimal place. **(6 marks)**

$2\sec^2 x + 7\sec x - 4 = 0$

$(2\sec x - 1)(\sec x + 4) = 0$

$\sec x = \frac{1}{2}$ ✗

$\sec x = -4$ ✓

$\frac{1}{\cos x} = -4$

$\cos x = -\frac{1}{4}$

$x = 104.47\ldots°, -104.47\ldots°, \cancel{255.52\ldots°}$

$x = 104.5°, -104.5°$ (1 d.p.)

Now try this

1 Solve, for $0 \leqslant x \leqslant 360°$,

$\cot^2 x + 2 = 3\cot x$

Give your answers in degrees to 1 decimal place. **(6 marks)**

Answer

2 Solve, for $0 \leqslant x \leqslant \pi$,

$\sqrt{3}\sec 2x = 2$

Give your answers in terms of π. **(5 marks)**

Answer