

10. The circle C_1 has Cartesian equation

$$x^2 + y^2 = 10x + k \quad x \in \mathbb{R} \quad y \in \mathbb{R}$$

where k is a constant.

The curve C_2 has parametric equations

$$x = t^2 \quad y = 2t \quad t \in \mathbb{R}$$

The curves C_1 and C_2 intersect at 4 distinct points.

Find the range of possible values for k , giving your answer in set notation.

(6)

At intersections,

$$\begin{aligned} x^2 + y^2 &= 10x + k \\ (t^2)^2 + (2t)^2 &= 10(t^2) + k \\ t^4 + 4t^2 &= 10t^2 + k \\ t^4 - 6t^2 - k &= 0 \\ (t^2)^2 - 6(t^2) - k &= 0 \end{aligned} \quad (2 \text{ marks})$$

$$\begin{aligned} \text{Roots must be real} &\Rightarrow b^2 - 4ac > 0 \\ (-6)^2 - 4(1)(-k) &> 0 \\ 36 + 4k &> 0 \\ k &> -9 \end{aligned} \quad (2 \text{ marks})$$

$$\text{Roots, } t^2, \text{ must also be positive } \frac{6 \pm \sqrt{36 + 4k}}{2} > 0$$

$$\text{Given } 36 + 4k > 0, \quad 6 + \sqrt{36 + 4k} > 0$$

we also need,

$$\begin{aligned} 6 - \sqrt{36 + 4k} &> 0 \\ \Rightarrow \frac{6 - \sqrt{36 + 4k}}{\sqrt{36 + 4k}} &< 6 \end{aligned}$$

$$\begin{aligned} &\text{because} \\ 36 + 4k > 0 &\Rightarrow 36 + 4k < 36 \Rightarrow 4k < 0 \Rightarrow k < 0 \quad (1 \text{ mark}) \end{aligned}$$

$$\text{Combining conditions in set notation: } \{k : k > -9\} \cap \{k : k < 0\} \quad (1 \text{ mark})$$