

13. (a) Express  $\frac{1}{x(100-x)}$  in partial fractions.

A scientist is studying the spread of a fungus in a field.  
The field has an area of  $100\text{m}^2$

The area of the field affected by the fungus,  $x\text{m}^2$ , is modelled by the differential equation

$$(a) \frac{1}{x(100-x)} \equiv \frac{A}{x} + \frac{B}{100-x}$$

$$500 \frac{dx}{dt} = x(100-x) \quad 1 \equiv A(100-x) + Bx$$

$$\text{when } x=100, 1=100B \Rightarrow B=0.01$$

where  $t$  days is the time after the start of the study.  $\text{when } x=0, 1=100A \Rightarrow A=0.01$  (1 mark)

Given that at the start of the study  $5\text{m}^2$  of the field was affected by the fungus,

(b) solve the differential equation to show that

$$\text{So, } \frac{1}{x(100-x)} = \frac{1}{100x} + \frac{1}{100(100-x)} \quad (1 \text{ mark})$$

(b) Separating the Variables,

$$500 \int \frac{dx}{x(100-x)} = \int dt$$

$$x = \frac{A}{1 + Be^{-\frac{1}{5}t}}$$

$$\text{From (a)} \Rightarrow \int \left( \frac{500}{100x} + \frac{500}{100(100-x)} \right) dx = \int dt$$

where  $A$  and  $B$  are constants to be found.

(6)

(c) Hence find the area of the field affected by the fungus 10 days after the start of the study.

(2)

$$(b) \text{ contd} \Rightarrow \int \left( \frac{5}{x} + \frac{5}{100-x} \right) dx = \int dt \quad (1 \text{ mark})$$

$$\Rightarrow 5 \ln x - 5 \ln(100-x) = t + c \quad (2 \text{ marks})$$

$$\text{Given } x=5 \text{ when } t=0, 5 \ln 5 - 5 \ln 95 = 0 + c$$

$$\Rightarrow c = 5 \ln \left( \frac{5}{95} \right) = 5 \ln \left( \frac{1}{19} \right) \quad (1 \text{ mark})$$

So,

$$5 \ln x - 5 \ln(100-x) = t + 5 \ln \left( \frac{1}{19} \right)$$

$$5 \ln \left( \frac{x}{100-x} \div \frac{1}{19} \right) = t \Rightarrow 5 \ln \left( \frac{19x}{100-x} \right) = t$$

$$\Rightarrow \ln \left( \frac{19x}{100-x} \right)^5 = t \Rightarrow e^t = \left( \frac{19x}{100-x} \right)^5 \Rightarrow e^{\frac{t}{5}} = \frac{19x}{100-x}$$

$$\Rightarrow 100e^{\frac{t}{5}} - xe^{\frac{t}{5}} = 19x \Rightarrow x(e^{\frac{t}{5}} + 19) = 100e^{\frac{t}{5}}$$

$$\Rightarrow x = \frac{100e^{\frac{t}{5}}}{e^{\frac{t}{5}} + 19} = \frac{100}{1 + 19e^{-\frac{t}{5}}} \quad (2 \text{ marks})$$

$$(c) \text{ when } t=10,$$

$$x = \frac{100}{1 + 19e^{-\frac{10}{5}}} = 28.000 \dots$$

$$= 28 \text{ m}^2 \text{ to nearest m}^2$$

(2 marks)